Announcements

Tardos OH today
1:45-2:45

- Last homework: **HW11** on computability due today

Section plans: week of May 4-5: quiz + review of material since the prelim

Final (cumulative): Saturday May 9th at 9am.

If you have a conflict, please fill out this form ASAP

Thursday, May 7, 3-7pm, in Hollister B14: final review led by a few TAs

Monday class: review of course & field of algorithms
Final info handout on canvas

TA application deadline has been extended: due May 10th

Divide and Conquer and convolution

Example n admitted students

student i: has prob p_i accepting offer, indep

Find for all k $\Pr(\#\text{ students accept} = k) = q_k$

Eg. $q_n = \prod_{i=1}^{n} p_i$ $\quad q_0 = \prod_{i=1}^{n} (1-p_i)$

$$q_k = \sum_{S \subseteq \{1,\dots,n\},\ |S|=k} \prod_{i \in S} p_i \prod_{j \notin S} (1-p_j)$$

2^n options to compute

Divide & conquer:

divide n people into two halves $n/2$ each

assume n power of 2

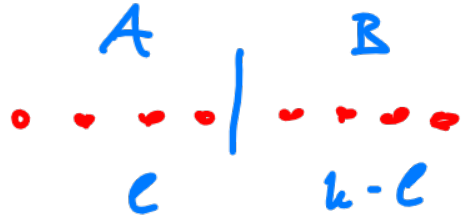
solve each group

q_k^A = probability that k accept in group A, q_k^B same for B

Simple divide and conquer method

given q_k^A & q_k^B all k can we combine?

$q_0^A\ q_1^A \cdots q_{n/2}^A \qquad q_0^B \cdots q_{n/2}^B$



$$q_k = \sum_{\ell=0}^{k} q_A^{\ell}\, q_B^{k-\ell}$$

all k
only $(n/2)^2$ products to compute
$O(n^2)$ time

$$T(n) = 2\,T(n/2) + c \cdot n^2$$

recursive call

↑ combine

$T(n)$ running time with n students

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What is the resulting running time using the method we have seen here so far?

$T(n) = 2T(n/2) + cn^2$

A. $O(n)$
B. $O(n \log n)$ ← top level already $O(n^2)$
C. $O(n^2)$ ← see master thm
D. $O(n^2 \log n)$ ← $O(n^2)$ at each level, $\log_2 n$ levels
E. More than any of these

Master thm:
$T(n) = qT(n/2) + cn^\alpha$
if $q < 2^\alpha \Rightarrow O(n^\alpha)$
$q = 2^\alpha \Rightarrow O(n \log n)$
$q > 2^\alpha \Rightarrow O(n^{\log_2 q})$

$n^2 + 2\left(\frac{n}{2}\right)^2 + 4\left(\frac{n}{4}\right)^2 \cdots$
$= n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \cdots \leq 2n^2$

Implementing merge more efficiently

$q_0^A \dots q_{n/2}^A \qquad q_0^B \dots q_{n/2}^B$

need to compute $q_k = \underbrace{\sum_{\ell=0}^{k} q_\ell^A \, q_{k-\ell}^B}$ convolution!

$O(n \log n)$ algorithm know FFT

Using this : $T(n) = 2T(n/2) + c \cdot n \log n$

Solving this: $c \cdot n \log n + c\cancel{2} \cdot \cancel{\frac{n}{2}} \log \frac{n}{2} + c\cancel{4} \cdot \cancel{\frac{n}{4}} \log \frac{n}{4} \leq c \cdot n \underbrace{\log n}_{\text{upper bound}} \cdot \underbrace{\log_2 n}_{\#\text{levels}}$

$O(n \log^2 n)$

[more precisely

$c\, n \log n + c\, n \log n/2 + c \cdot n \log n/4 + \dots = c \cdot n (\log n)^2/2$]

Examples of Convolutions

you have seen

1. $f(x) = a_0 + a_1 x + \cdots a_n x^n \qquad g(x) = b_0 + b_1 x + \cdots + b_n x^n$

$f(x)\,g(x) = a_0 b_0 + \underbrace{(a_0 b_1 + a_1 b_0)}_{\text{convolution}} x + \cdots$

$O(n \log n)$ time

2. X & Y indep

$Pr(X = k) = p_k^X \qquad Pr(Y = k) = p_k^Y$

$$Pr(X + Y = k) = \sum_{\ell=0}^{k} Pr(X = \ell)\, Pr(Y = k - \ell) = \underbrace{\sum_{\ell=0}^{k} p_\ell^X\, p_{k-\ell}^Y}_{\text{convolution}}$$

Summary of what you need to know about divide and conquer

- divide into small groups (usually 2 equal size)
- solve parts recursively
- figure out how to combine ← algo question $O(n^\alpha)$

$\Rightarrow$ overall time $T(n) = 2\,T(n/2) + c \cdot n^\alpha$

use master thm to get resulting run time

alternately add times used at recursive levels

$cn^\alpha + 2 \cdot c\left(\frac{n}{2}\right)^\alpha + 4c\left(\frac{n}{4}\right)^\alpha + \cdots \quad 2^{\log_2 n}\, c\left(\frac{n}{2^{\log_2 n}}\right)^\alpha$

recall: $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$

- $a > 1 \rightsquigarrow O(a^k)$
- $a = 1$ k
- $a < 1 \rightsquigarrow \frac{1}{1-a}$

Summary of what you need to know about divide and conquer

Quick find

$a_1 \ldots\ldots a_n$ find k^{th} in time faster than sorting

Pick a_i at random

$|S_-|, |S_+| \geq n/4$

find a_i with this

Expected $O(n)$ comparisons

$T(n) = 1 \cdot T(\frac{3}{4}n) + c \cdot n$ expected run time

$c \cdot n + c \cdot \frac{3}{4}n + c\left(\frac{3}{4}\right)^2 \cdot n = O(n)$ $\left[c \cdot \frac{1}{1-3/4} \cdot n\right]$